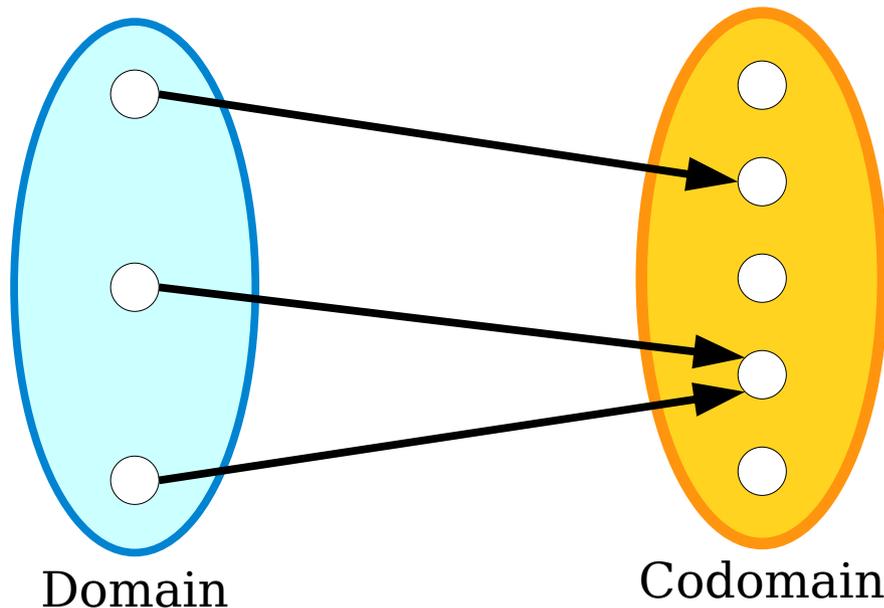Lecture 07:
Functions

Part 2 of 2

Functions

Part 2

1. **Recap from Last Time**
2. A Proof About Birds
3. Assuming vs. Proving
4. Leveling Up the Assume-Prove Table
5. Connecting Function Types
6. The Two-Column Proof Organizer
7. Announcements
8. Function Composition
9. Properties of Composition: A Proof
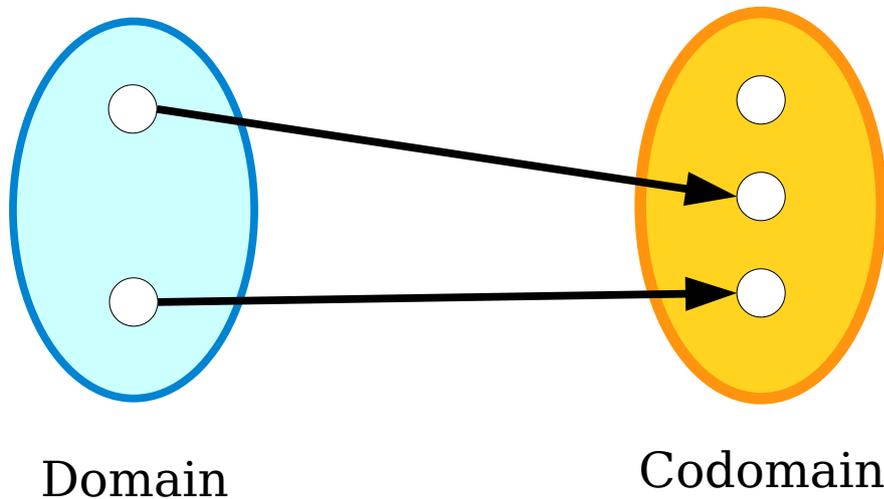10. Major Ideas from Today
11. What's Next?

Recap from Last Time

Is it a *function?* Yes!

Is it an *injection?* No.

Is it a *surjection?* No.

Domain

Codomain

Recap from Last Time

Is it a *function?*   Yes!

Is it an *injection?* Yes!

Is it a *surjection?* No.

Recap from Last Time

Domain

Codomain

*Is it a **function**?*   **Yes!**

*Is it an **injection**?*   **No.**

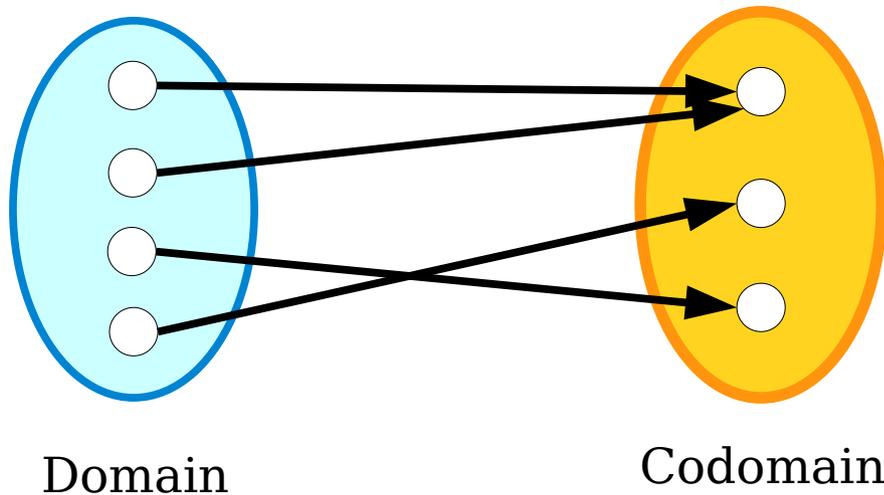*Is it a **surjection**?*   **Yes!**

Recap from Last Time

Injection: $\forall a_1 \in A. \ \forall a_2 \in A. \ (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2))$

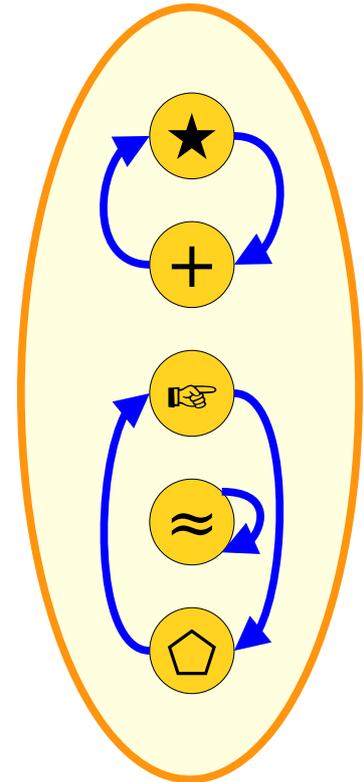*If the inputs are different, the outputs are different*

- Can also define with the contrapositive!

Surjection: $\forall b \in B. \ \exists a \in A. \ f(a) = b$

"For every possible output, there's an input that produces it."

Involution: $\forall x \in A. \ f(f(x)) = x$

"Applying f twice is equivalent to not applying f at all."

The Assume-Prove Table

		To **prove** that this is true…
$\forall x.\ A$		Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$		Find an x where A is true. Then prove A is true for that specific choice of x.
$A \rightarrow B$		Assume A is true, then prove B is true.
$A \wedge B$		Prove A. Also prove B.
$A \vee B$		Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$. *(Why does this work?)*
$A \leftrightarrow B$		Prove $A \rightarrow B$ and $B \rightarrow A$.
$\neg A$		Simplify the negation, then consult this table on the result.

Functions

Part 2
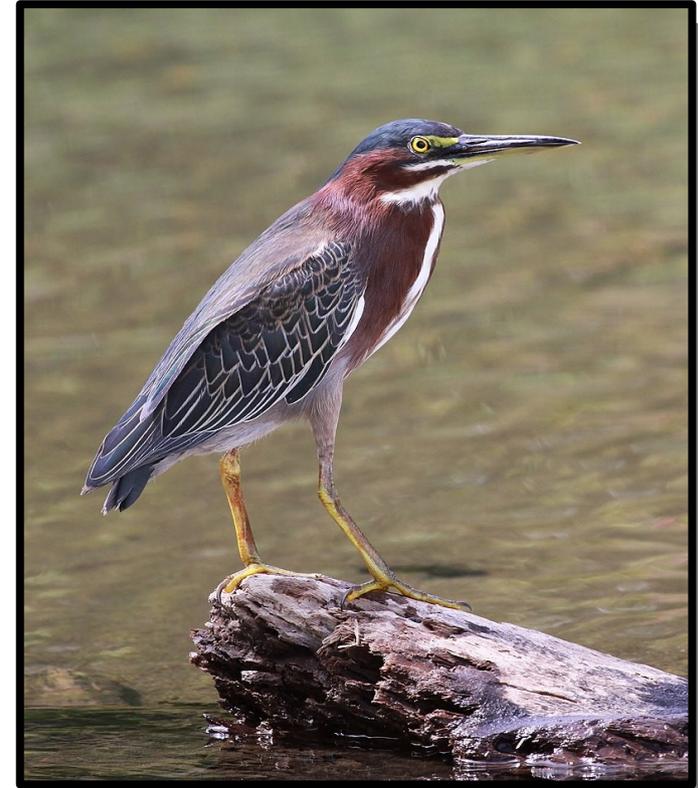
Theorem: If all birds have feathers,
then all herons have feathers.

A Proof About Birds

Theorem: If all birds have feathers, then all herons have feathers.

Given the predicates

    Bird(_b_), which says _b_ is a bird;
    Heron(_h_), which says _h_ is a heron; and
    Feathers(_x_), which says _x_ has feathers,

translate the theorem into first-order logic.

$$(\forall b.\,(Bird(b) \to Feathers(b))) \to (\forall h.\,(Heron(h) \to Feathers(h)))$$

All birds have feathers.

All herons have feathers.

A Proof About Birds

Theorem: If all birds have feathers, then all herons have feathers.

Proof: Assume that all birds have feathers. We will show that all herons have feathers.
Consider an arbitrary bird b. Since b is a bird, b has feathers. *[And now we're stuck! We are interested in herons, but b might not be one. It could be a hummingbird, for example!]*

$$(\forall b.\,(Bird(b) \rightarrow Feathers(b))) \rightarrow (\forall h.\,(Heron(h) \rightarrow Feathers(h)))$$
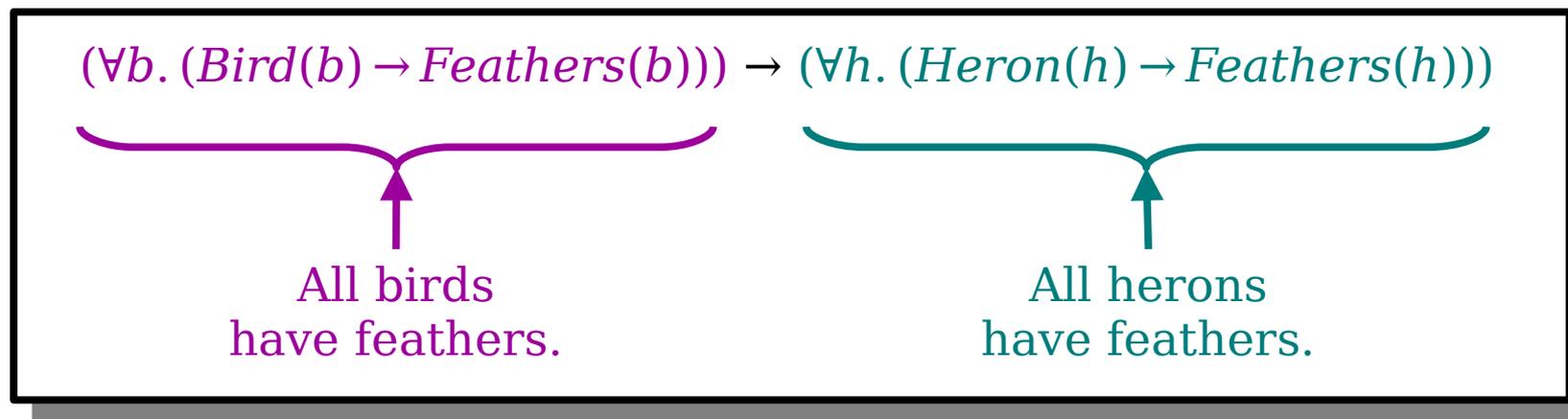
All birds have feathers.

All herons have feathers.

A Proof About Birds

Theorem: If all birds have feathers, then all herons have feathers.

Proof: Assume that all birds have feathers. We will show that all herons have feathers.

Consider an arbitrary heron h. We will show that h has feathers. To do so, note that since h is a heron we know h is a bird. Therefore, by our earlier assumption, h has feathers. ∎
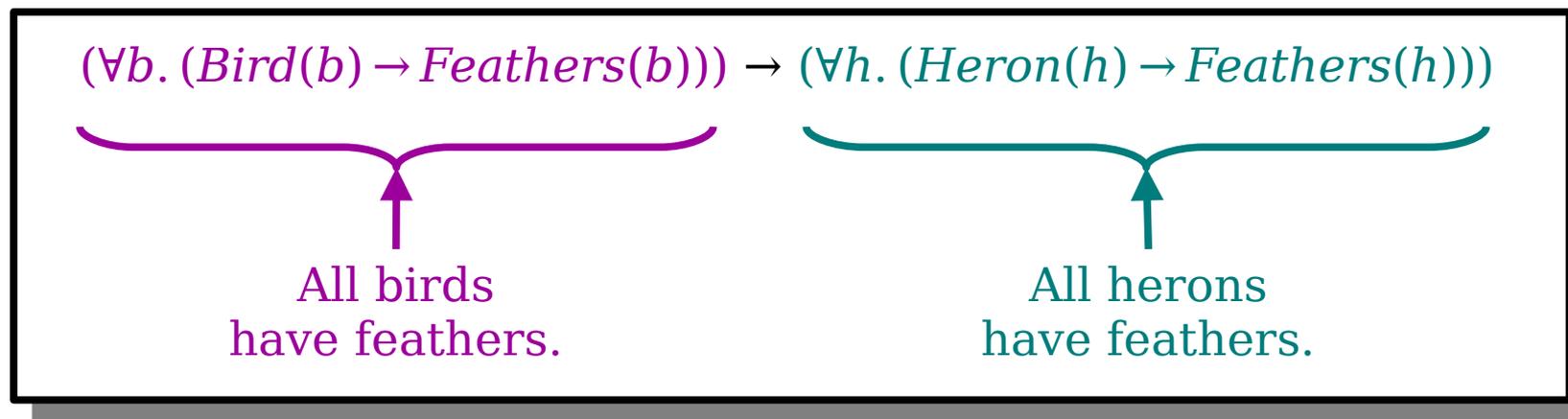
$$(\forall b.\, (Bird(b) \to Feathers(b))) \to (\forall h.\, (Heron(h) \to Feathers(h)))$$

All birds have feathers.

All herons have feathers.

Functions

Part 2

Proving vs. Assuming

- In the context of a proof, you will need to assume some statements and prove others.

  - Here, we **_assumed_** all birds have feathers.

  - Here, we **_proved_** all herons have feathers.

- Statements behave differently based on whether you're assuming or proving them.

$$(\forall b.\,(Bird(b) \rightarrow Feathers(b))) \rightarrow (\forall h.\,(Heron(h) \rightarrow Feathers(h)))$$

We never introduce a variable b.

We introduce a variable h almost immediately.

Proving vs. Assuming

- To ***prove*** the universally-quantified statement

$$\forall x.\ P(x)$$

  we introduce a new variable x representing some arbitrarily-chosen value.

- Then, we prove that $P(x)$ is true for that variable x.

- That's why we introduced a variable h in this proof representing a heron.

$$(\forall b.\ (Bird(b) \rightarrow Feathers(b))) \rightarrow (\forall h.\ (Heron(h) \rightarrow Feathers(h)))$$

We never introduce a variable b.

We introduce a variable h almost immediately.

Proving vs. Assuming

- If we ***assume*** the statement

$$\forall x.\ P(x)$$

  we ***do not*** introduce a variable x.

- Rather, if we find a relevant value z somewhere else in the proof, we can conclude that $P(z)$ is true.

- That's why we didn't introduce a variable b in our proof, and why we concluded that h, our heron, have feathers.

$$(\forall b.\ (Bird(b) \rightarrow Feathers(b))) \rightarrow (\forall h.\ (Heron(h) \rightarrow Feathers(h)))$$

We never introduce a variable b.

We introduce a variable h almost immediately.

Functions

Part 2

The Assume-Prove Table

	If you **assume** this is true...	To **prove** that this is true...
$\forall x.\ A$	Initially, **do nothing**. Once you find a z through other means, you can state it has property A.	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$	Introduce a variable x into your proof that has property A.	Find an x where A is true. Then prove A is true for that specific choice of x.
$A \rightarrow B$	Initially, **do nothing**. Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \wedge B$	Assume A. Also assume B.	Prove A. Also prove B.
$A \vee B$	Consider two cases. Case 1: A is true. Case 2: B is true.	Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$. *(Why does this work?)*
$A \leftrightarrow B$	Assume $A \rightarrow B$ and $B \rightarrow A$.	Prove $A \rightarrow B$ and $B \rightarrow A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Functions

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Types of Functions

- We now have three special types of functions:

  - *involutions*, functions that undo themselves;

  - *injections*, functions where different inputs go to different outputs; and

  - *surjections*, functions that cover their whole codomain.

- *Question:* How do these three classes of functions relate to one another?

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

$$(\forall x \in A.\ f(f(x)) = x) \quad \rightarrow \quad (\forall b \in A.\ \exists a \in A.\ f(a) = b)$$

f is an involution.

f is surjective.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$$(\forall x \in A.\ f(f(x)) = x) \rightarrow (\forall b \in A.\ \exists a \in A.\ f(a) = b)$$

Assume this.

Prove this.

If you **assume** this is true...
Initially, **do nothing**. Once you find a z through other means, you can state it has property A.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$$(\forall x \in A.\ f(f(x)) = x) \quad \rightarrow \quad (\forall b \in A.\ \exists a \in A.\ f(a) = b)$$

Assume this.

Prove this.

Since we're *assuming* this, we aren't going to pick a specific choice of x right now. Instead, we're going to keep an eye out for something to apply this fact to.

Proof Outline

1. Assume f is an involution.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$(\forall x \in A. \; f(f(x)) = x) \quad \rightarrow \quad (\forall b \in A. \; \exists a \in A. \; f(a) = b)$

Ass

There's a universal quantifier up front. Since we're *proving* this, we'll pick an arbitrary $b \in A$.

Prove this.

Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$$(\forall x \in A.\ f(f(x)) = x) \quad \rightarrow \quad (\forall b \in A.\ \exists a \in A.\ f(a) = b)$$
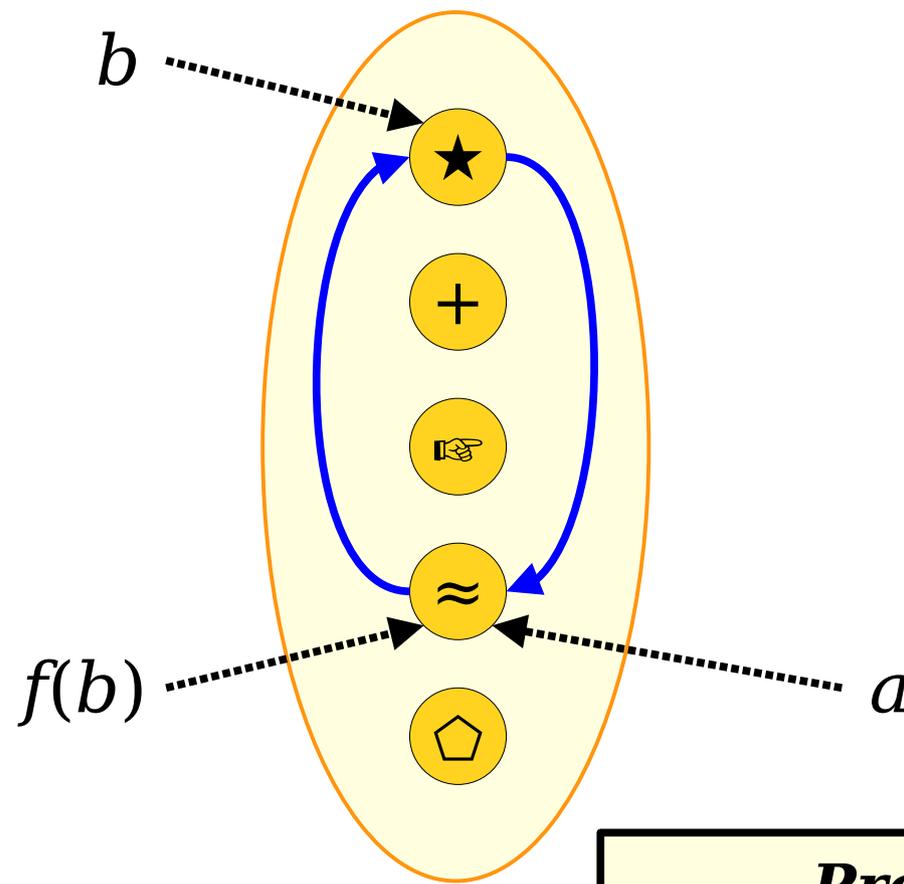
Prove this.

Now, we hit an existential quantifier. Since we're **proving** this, we need to find a choice of $a \in A$ where this is true.

Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.
3. Give a choice of $a \in A$ where $f(a) = b$.

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.
3. Give a choice of $a \in A$ where $f(a) = b$.

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

Proof: Pick any involution $f : A \to A$. We will prove that f is surjective. To do so, pick an arbitrary $b \in A$. We need to show that there is an $a \in A$ where $f(a) = b$.

Specifically, pick $a = f(b)$. This means that $f(a) = f(f(b))$, and since f is an involution we know that $f(f(b)) = b$. Putting this together, we see that $f(a) = b$, which is what we needed to show. ∎

Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.
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Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

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Specifically, pick $a = f(b)$. This means that $f(a) = f(f(b))$, and since f is an involution we know that $f(f(b)) = b$. Putting this together, we see that $f(a) = b$, which is what we needed to show. ∎

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

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Theorem: Let $f : A \to A$ be an involution. Then f is injective.

Theorem: Let $f : A \to A$ be an involution.
Then f is injective.

What We're Assuming	*What We Need to Prove*

$f : A \to A$ is an involution.

$\forall z \in A.\ f(f(z)) = z.$

f is injective.

$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to$
$\qquad a_1 = a_2$
$)$

We're *assuming* this universally-quantified statement, so we won't introduce a variable for what's here.

We need to *prove* this universally-quantified statement. So let's introduce arbitrarily-chosen values.

Theorem: Let $f : A \to A$ be an involution. Then f is injective.

What We're Assuming	What We Need to Prove
$f : A \to A$ is an involution.	f is injective.
$\forall z \in A.\ f(f(z)) = z.$	~~$\forall a_1 \in A.\ \forall a_2 \in A.$~~ $(f(a_1) = f(a_2) \to$
	$a_1 = a_2$
$a_1 \in A$	$)$
$a_2 \in A$	

Theorem: Let $f : A \to A$ be an involution. Then f is injective.

What We're Assuming	*What We Need to Prove*
$f : A \to A$ is an involution.	f is injective.

What We're Assuming

$f : A \to A$ is an involution.

$\quad \forall z \in A.\ f(f(z)) = z.$

$a_1 \in A$

$a_2 \in A$

What We Need to Prove

f is injective.

$\quad \cancel{\forall a_1 \in A.\ \forall a_2 \in A.}\ (f(a_1) = f(a_2) \to$

$\qquad a_1 = a_2$

$\quad)$

We need to prove
this **implication**. So
we **assume the antecedent**
and **prove the consequent.**

Theorem: Let $f : A \to A$ be an involution.
Then f is injective.

What We're Assuming	*What We Need to Prove*
$f : A \to A$ is an involution.	f is injective.
$\quad \forall z \in A.\ f(f(z)) = z.$	~~$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to$~~
	$\quad\quad a_1 = a_2$
$a_1 \in A$	~~$)$~~

$a_2 \in A$

$f(a_1) = f(a_2)$

$f(f(a_1)) = f(f(a_2))$
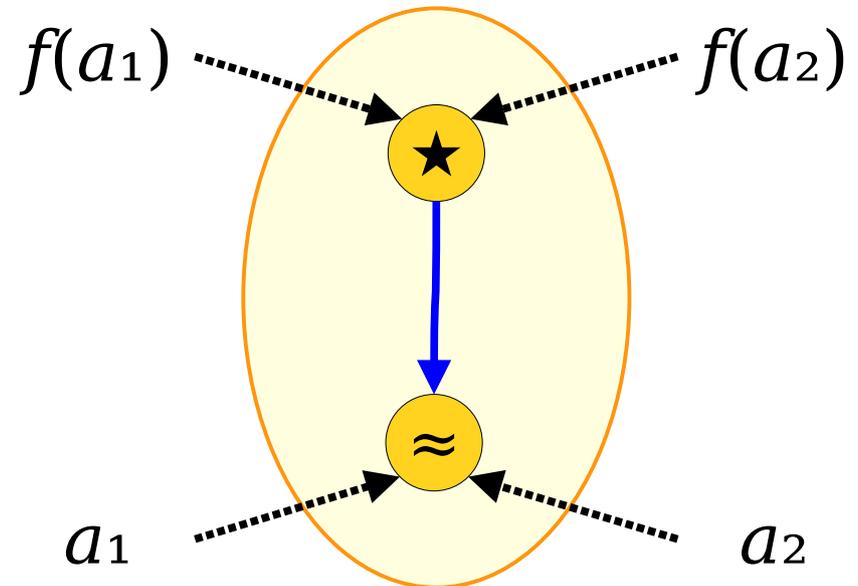
$f(f(a_1)) = a_1$

$f(f(a_2)) = a_2$

Theorem: Let $f : A \to A$ be an involution. Then f is injective.

Proof: Choose any $a_1, a_2 \in A$ where $f(a_1) = f(a_2)$. We need to show that $a_1 = a_2$.

Since $f(a_1) = f(a_2)$, we know that $f(f(a_1)) = f(f(a_2))$. Because f is an involution, we see $a_1 = f(f(a_1))$ and that $f(f(a_2)) = a_2$. Putting this together, we see that

$$a_1 = f(f(a_1)) = f(f(a_2)) = a_2,$$

so $a_1 = a_2$, as needed. ∎

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

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Announcements

- ***Problem Set Three***
  - posted today, due next Friday at 1:00 PM
  - some parts require Monday's lecture; clearly labeled
- ***Problem Set Two***
  - can use a late day to submit by Saturday at 1:00 PM
- ***Left-handed desk form***
  - due Monday of next week
- ***Attendance opt-out form***
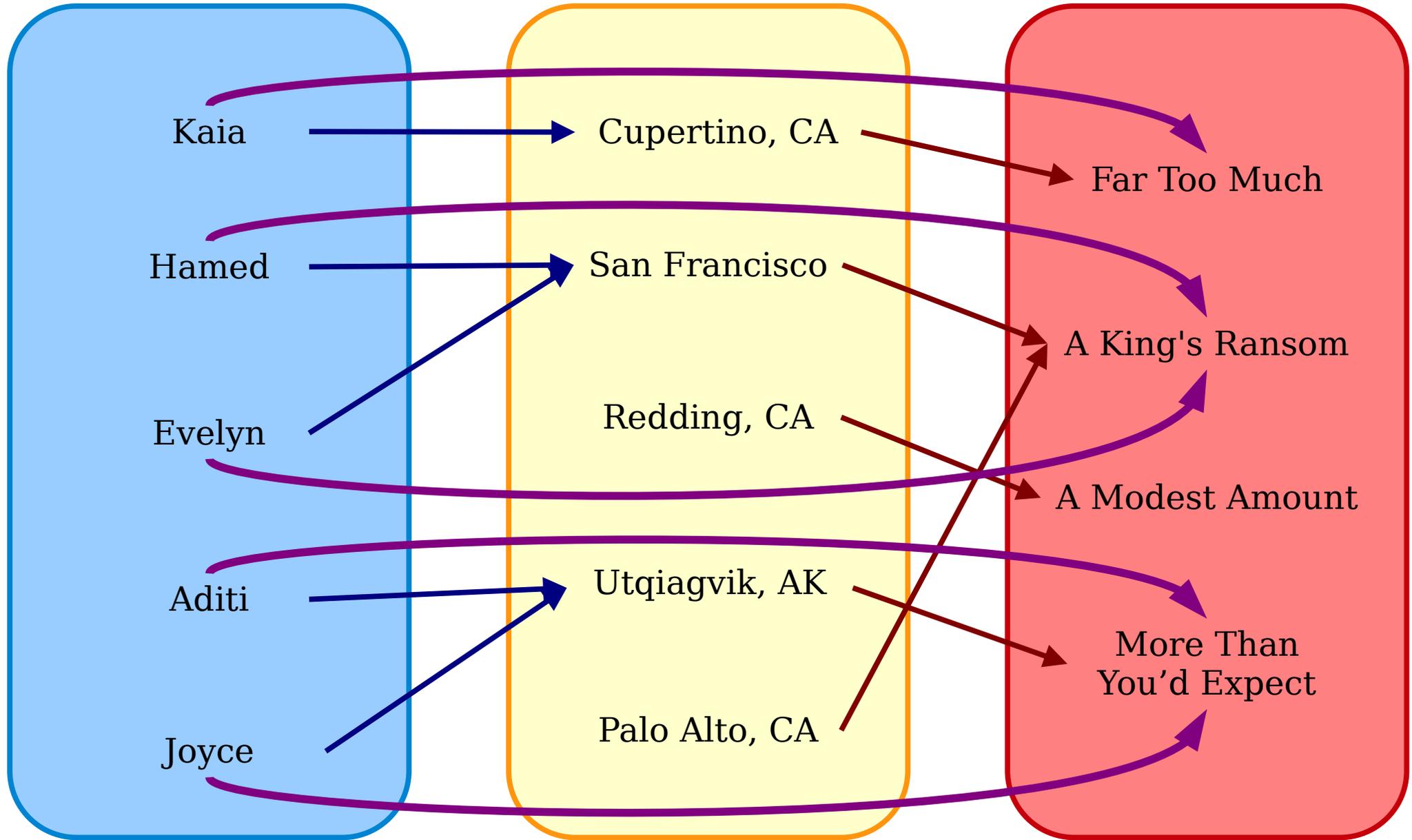 - available sometime next Monday, due next Friday

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f : People → Places

g : Places → Prices

Kaia

Cupertino, CA

Far Too Much

Hamed

San Francisco

A King's Ransom

Evelyn

Redding, CA

A Modest Amount

Aditi

Utqiagvik, AK

More Than
You'd Expect
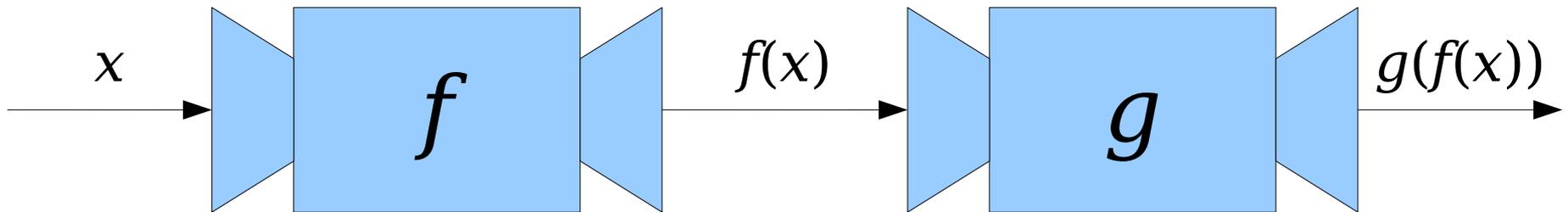
Joyce

Palo Alto, CA

People

Places

Prices

h : People → Prices
h(x) = g(f(x))

Function Composition

- Suppose that we have two functions $f : A \rightarrow B$ and $g : B \rightarrow C$.

- Notice that the codomain of f is the domain of g. This means that we can use outputs from f as inputs to g.

Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.

- The ***composition of f and g***, denoted ***g ∘ f***, is a function where

  - $g \circ f : A \to C$, and

  - $(g \circ f)(x) = g(f(x))$.

- A few things to notice:

> The name of the function is $g \circ f$. When we apply it to an input x, we write $(g \circ f)(x)$. I don't know why, but that's what we do.

  - The domain of $g \circ f$ is the domain of f. Its codomain is g's codomain.

  - Even though composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.

- Composition is ***associative***: $(f \circ g) \circ h = f \circ (g \circ h)$. (Prove this!)

- Composition is not necessarily commutative: $f \circ g$ is not necessarily the same as $g \circ f$. (Prove this!)

Functions

Part 2

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	*What We Need to Prove*

What We're Assuming

$f : A \to B$ is an injection.

$\forall x \in A. \ \forall y \in A. \ (x \neq y \to f(x) \neq f(y)$
)

$g : B \to C$ is an injection.

$\forall x \in B. \ \forall y \in B. \ (x \neq y \to g(x) \neq g(y)$
)

We're **assuming** these universally-quantified statements, so we won't introduce any variables for what's here.

What We Need to Prove

$g \circ f$ is an injection.

$\forall a_1 \in A. \ \forall a_2 \in A. \ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2)$
)

We need to **prove** this universally-quantified statement. So let's introduce arbitrarily-chosen values.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	***What We Need to Prove***
$f : A \to B$ is an injection.	$g \circ f$ is an injection.

What We're Assuming

$f : A \to B$ is an injection.

$\forall x \in A. \ \forall y \in A. \ (x \neq y \to$
$\quad f(x) \neq f(y)$
$)$

$g : B \to C$ is an injection.

$\forall x \in B. \ \forall y \in B. \ (x \neq y \to$
$\quad g(x) \neq g(y)$
$)$

$a_1 \in A$ is arbitrarily-chosen.

$a_2 \in A$ is arbitrarily-chosen.

What We Need to Prove

$g \circ f$ is an injection.

$\forall a_1 \in A. \ \forall a_2 \in A. \ (a_1 \neq a_2 \to$
$\quad (g \circ f)(a_1) \neq (g \circ f)(a_2)$
$)$

> We need to **prove** this universally-quantified statement. So let's introduce arbitrarily-chosen values.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	**What We Need to Prove**
$f : A \to B$ is an injection.	$g \circ f$ is an injection.
$\forall x \in A. \; \forall y \in A. \; (x \neq y \to f(x) \neq f(y)$ $)$	$\forall a_1 \in A. \; \forall a_2 \in A. \; (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2)$ $)$
$g : B \to C$ is an injection.	
$\forall x \in B. \; \forall y \in B. \; (x \neq y \to g(x) \neq g(y)$ $)$	Now we're looking at an implication. Let's **assume** the antecedent and **prove** the consequent.
$a_1 \in A$ is arbitrarily-chosen.	
$a_2 \in A$ is arbitrarily-chosen.	
$a_1 \neq a_2$	

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	What We Need to Prove
$f : A \to B$ is an injection. $\quad \forall x \in A. \ \forall y \in A. \ (x \neq y \to$ $\qquad f(x) \neq f(y)$ $\quad)$ $g : B \to C$ is an injection. $\quad \forall x \in B. \ \forall y \in B. \ (x \neq y \to$ $\qquad g(x) \neq g(y)$ $\quad)$ $a_1 \in A$ is arbitrarily-chosen. $a_2 \in A$ is arbitrarily-chosen. $a_1 \neq a_2$	$g \circ f$ is an injection. $\quad \forall a_1 \in A. \ \forall a_2 \in A. \ (a_1 \neq a_2 \to$ $\qquad (g \circ f)(a_1) \neq (g \circ f)(a_2)$ $\quad)$ Let's write this out separately and simplify things a bit.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

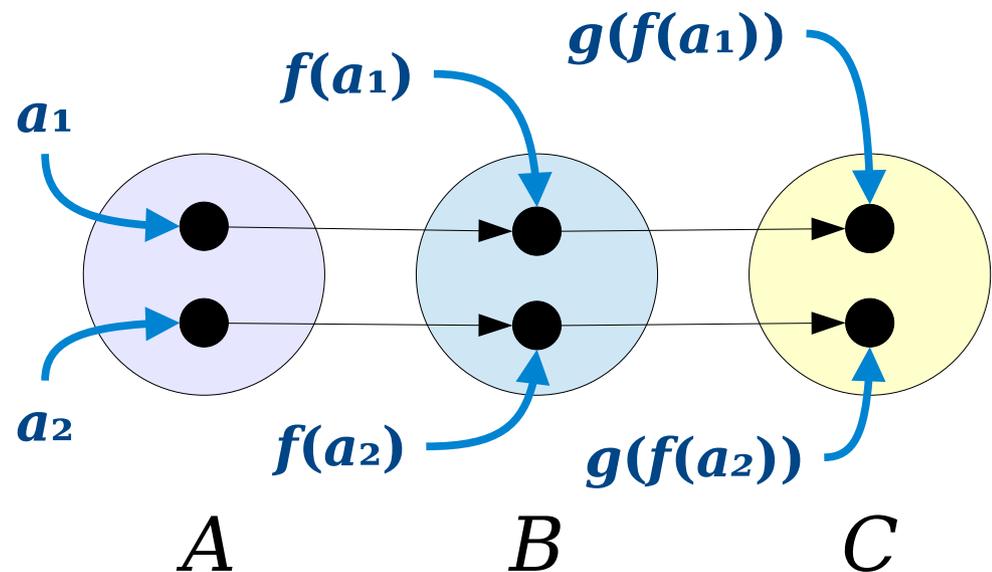| What We're Assuming	What We Need to Prove
$f : A \to B$ is an injection.	$g \circ f$ is an injection.
$\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$	$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$
$g : B \to C$ is an injection.	$g(f(a_1)) \neq g(f(a_2))$
$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$	
$a_1 \in A$ is arbitrarily-chosen.	
$a_2 \in A$ is arbitrarily-chosen.	
$a_1 \neq a_2$	

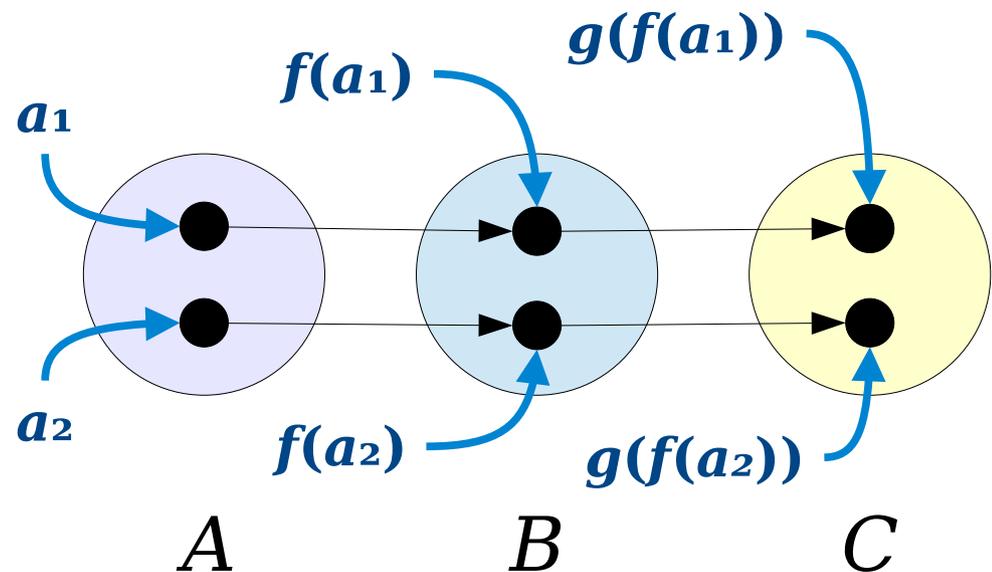***Theorem:*** If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.

Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ∎

Great exercise: Repeat this proof using the other definition of injectivity.
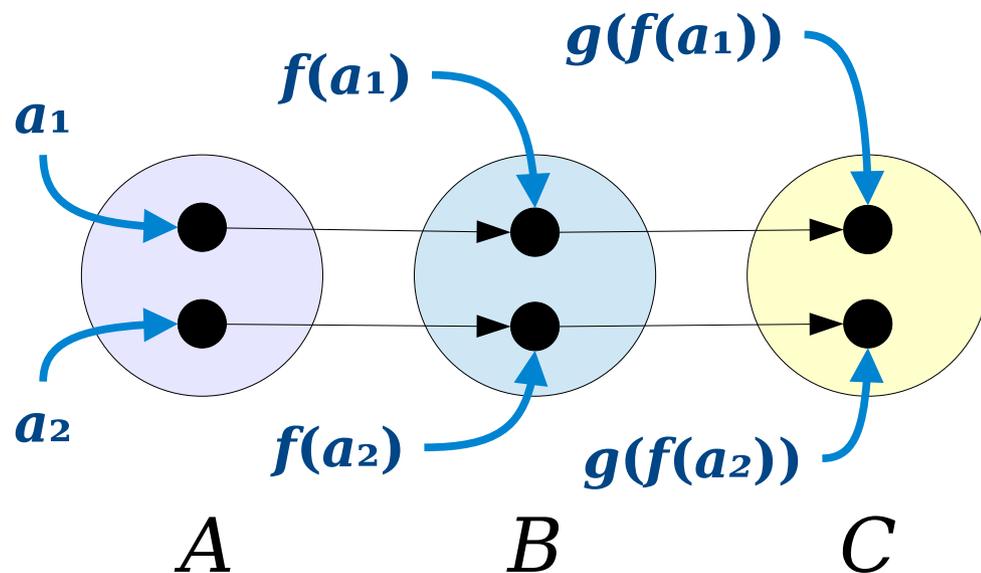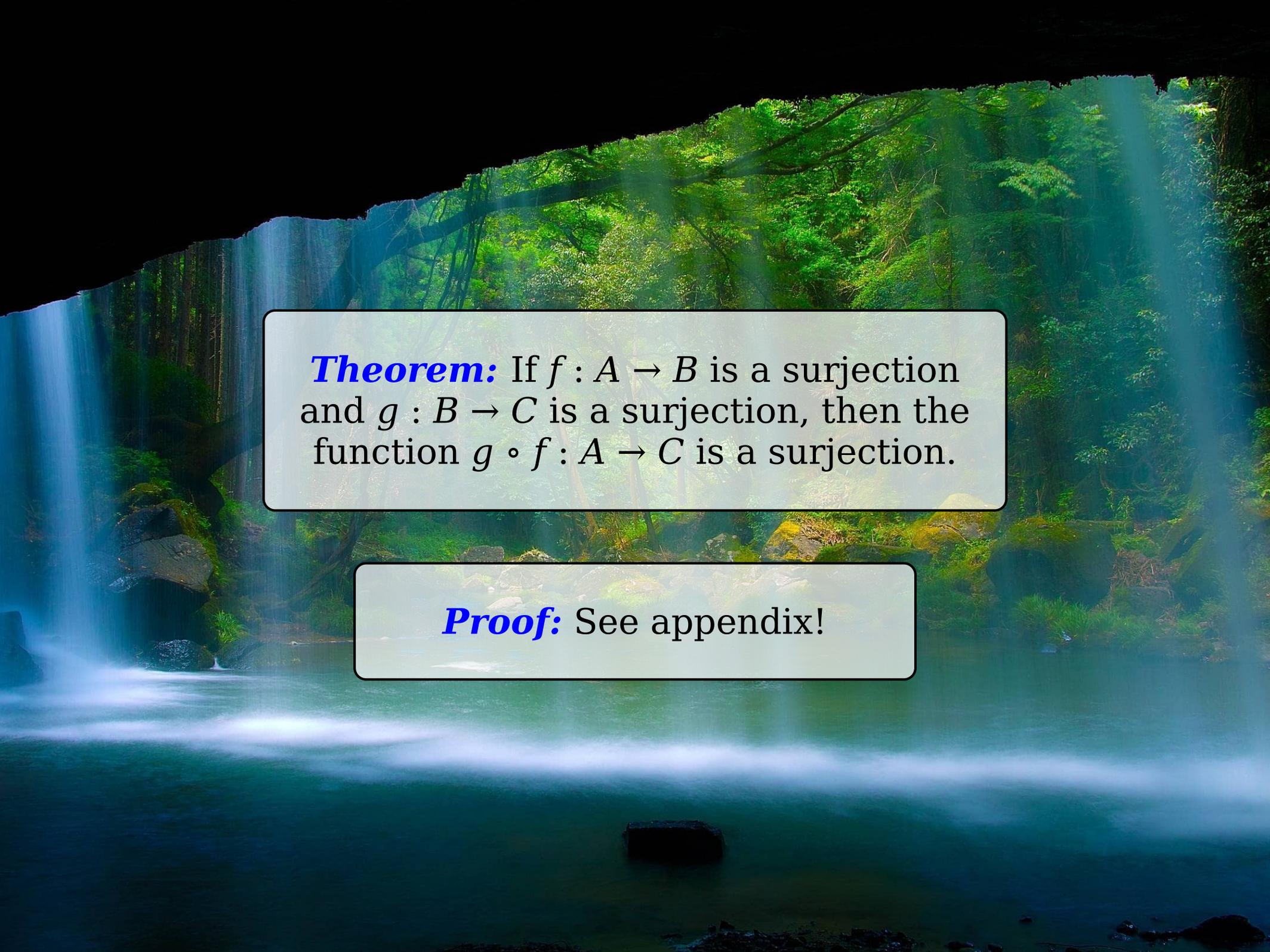
Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.

Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ∎

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

Theorem: If $f : A \to B$ is a surjection and $g : B \to C$ is a surjection, then the function $g \circ f : A \to C$ is a surjection.

Proof: See appendix!

Functions

Part 2

1. Recap from Last Time
2. A Proof About Birds
3. Assuming vs. Proving
4. Leveling Up the Assume-Prove Table
5. Connecting Function Types
6. The Two-Column Proof Organizer
7. Announcements
8. Function Composition
9. Properties of Composition: A Proof
10. **Major Ideas from Today**
11. What's Next?

Major Ideas From Today

- Proofs involving first-order definitions are heavily based on the structure of those definitions, yet FOL notation itself does *not* appear in the proof.

- Statements behave differently based on whether you're *assuming* or *proving* them.

- When you *assume* a universally-quantified statement, initially, do nothing. Instead, keep an eye out for a place to apply the statement more specifically.

- When you *prove* a universally-quantified statement, pick an arbitrary value and try to prove it has the needed property.

The Assume-Prove Table

	If you ***assume*** this is true...	To ***prove*** that this is true...
$\forall x.\ A$	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$	Introduce a variable x into your proof that has property A.	Find an x where A is true. Then prove A is true for that specific choice of x.
$A \rightarrow B$	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \wedge B$	Assume A. Also assume B.	Prove A. Also prove B.
$A \vee B$	Consider two cases. Case 1: A is true. Case 2: B is true.	Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$. *(Why does this work?)*
$A \leftrightarrow B$	Assume $A \rightarrow B$ and $B \rightarrow A$.	Prove $A \rightarrow B$ and $B \rightarrow A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

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What's Next?

- ***Set Theory Revisited***

  - Formalizing our definitions.

- ***Proofs on Sets***

  - How to rigorously establish set-theoretic results.

Appendix:
Additional Function Proofs

Proof: Composing surjections yields a surjection.

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$. Equivalently, we will prove that for any $c \in C$, there is some $a \in A$ such that $g(f(a)) = c$.

Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \to B$ is surjective, there is some $a \in A$ such that $f(a) = b$. Then we see that

$$g(f(a)) = g(b) = c,$$

which is what we needed to show. ∎